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Representation of the Yangian invariant motif and the Macdonald polynomial

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Abstract. The representation of the Yangian invariant 'motif' is considered. The relationship with the Rogers–Szegö polynomial is studied, whose one-parameter deformation is the Macdonald polynomial. We propose the deformation of the motifs which provides a new realization of the Macdonald polynomials for the one-row Young diagrams.

1. Introduction

The Yangian symmetry has appeared in recent studies in statistical mechanics and mathematical physics. In these studies, the long-range interacting su(n) spin chain called the Haldane–Shastry (HS) model [1, 2] plays an important role. The Hamiltonian of the HS model is defined as

$$\mathcal{H}^{\mathrm{HS}} = \sum_{1\leqslant j<k\leqslant N} \frac{z_j\, z_k}{(z_j - z_k)\,(z_k - z_j)} P_{jk} \tag{1.1}$$

where $z_k = \exp(2\pi \mathrm{i} k/N)$ is a coordinate for the kth spin and the operator P_{jk} permutes the jth and kth su(n) spin states. It was shown [3, 4] that the Hamiltonian $\mathcal{H}^{\mathrm{HS}}$ reveals the Yangian symmetry even for a finite number of sites N. Based on this fact, Haldane *et al* [3] proposed the Yangian invariant basis 'motif' to classify the energy spectrum and the degeneracy.

Another important long-ranged interacting su(n) spin chain is the Polychronakos–Frahm (PF) model [5, 6], whose Hamiltonian is given by

$$\mathcal{H}^{\mathrm{PF}} = \sum_{1\leqslant j<k\leqslant N} \frac{1}{(z_j - z_k)^2} P_{jk}. \tag{1.2}$$

Here the position of the jth spin z_j is the zeroth value of the Nth Hermite polynomial, i.e. the equilibrium point of the Calogero model confined in the external harmonic potential, and the system is not translationally invariant. It was shown [7, 8] that the N-spin PF model also exactly possesses the Yangian symmetry, and that their eigenstates can also be written by the motifs. It is of interest that the energy spectrum of the PF model is equally spaced and that we can calculate the partition function $Z_N(x; q)$ for an N-spin chain. This fact reminds us that the Yangian symmetry is realized in terms of currents of the level-1 su(n) WZNW theory and that the Virasoro generator L_0 is the lowest conserved operator [9]. In

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fact, the Hamiltonian $\mathcal{H}^{\mathrm{PF}}$ was shown [7] to correspond to the Virasoro generator L_0, and that the partition function $Z_N(x;q)$ reduces in the large-N limit to the affine character of the su$(n)_1$ WZNW theory,

$$\mathrm{ch}_{\Lambda_\ell}(x;q) = \lim_{\substack{N\to\infty \\ N\equiv\ell \pmod n}} Z_N(x;q).$$

As the result of this close relationship between the Yangian symmetry and the character formula, the representation for the motif becomes significant. Haldane *et al* [3, 4] established the representation for the su(2) motifs, but the su(n) motif was not completed. The first attempt followed from the fact that the partition function $Z_N(x;q)$ of the su(n) PF model is related to the generalized Rogers–Szegö (RS) polynomial; based on the recurrence relation of the RS polynomial, the representations for the su(n) motifs are established in the recurrence formulae [7]. Recently two groups [10, 11] have studied the representation for the su(n) motifs from different points of view, the conformal field theory and the RSOS model.

In this paper, we reconsider the representation of the Yangian invariant su(n) motif. We stress that the representation is closely related to the RS polynomials. As the RS polynomial is a degenerate case of the Macdonald polynomial for a one-row Young diagrams, we shall deform the representation of the Yangian invariant motif. We investigate the role of parameters q and t in the Macdonald polynomials using this approach.

2. Rogers–Szegö polynomial

The su(n) PF model $\mathcal{H}^{\mathrm{PF}}$ has the Yangian symmetry Y(su(n)) as in the case of the HS model $\mathcal{H}^{\mathrm{HS}}$ [7]. The stimulating fact is that the energy spectrum of the PF model is equally spaced, and that we can exactly calculate its partition function $Z_N(x;q) = \mathrm{Tr}\, q^{\mathcal{H}^{\mathrm{PF}}}$ [5]. The explicit form of the partition function is given by

$$Z_N(x;q) = q^{[(n-1)/2n]N^2} H_N(x;q^{-1}). \tag{2.1}$$

Here $H_N(x;q) = H_N(x_1,\ldots,x_n;q)$ is called the generalized RS polynomial [12, 13],

$$H_N(x;q) = \sum_{\substack{k_1+\cdots+k_n=N \\ k_j\geqslant 0}} \begin{bmatrix} N \\ k_1,k_2,\ldots,k_n \end{bmatrix}_q x_1^{k_1} x_2^{k_2}\ldots x_n^{k_n} \tag{2.2}$$

where the q-multinomial polynomial is defined as

$$\begin{bmatrix} N \\ k_1,k_2,\ldots,k_n \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_N}{(q;q)_{k_1}\ldots(q;q)_{k_n}} & \text{for } k_1+\cdots+k_n = N \text{ and } k_j \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $(x;q)_k$ denotes the q-product,

$$(x;q)_k = (1-x)(1-x\,q)\ldots(1-x\,q^{k-1}) \qquad (x;q)_0 = 1.$$

We note in appendix A explicit forms of the RS polynomials $H_N(x;q)$ up to $N=5$ in terms of the Schur function $s_\lambda(x)$. The RS polynomial may be viewed as the q-deformation of the Hermite polynomials and its generating function is given as follows

$$\sum_{N=0}^{\infty} \frac{H_N(x;q)}{(q;q)_N} z^N = \prod_{j=1}^{n} \frac{1}{(x_j z;q)_\infty}. \tag{2.3}$$

By setting $z \to qz$ in the above identity we obtain the recurrence relation for any $(n+1)$-consecutive RS polynomials

$$H_N(x;q) = \sum_{k=1}^{n} (-)^{k-1} \frac{(q;q)_{N-1}}{(q;q)_{N-k}} e_k(x) H_{N-k}(x;q) \tag{2.4}$$

where $e_k(x)$ is the elementary symmetric function, in terms of the Schur function $s_\lambda(x)$, the elementary symmetric function $e_k(x)$ is written as

$$e_k(x) = s_{[1^k]}(x).$$

The recurrence relation (2.4) and the initial conditions, $H_N(x;q) = 0$ for $N < 0$ and $H_0(x;q) = 1$, uniquely determine the q-polynomial $H_N(x;q)$.

It was proved that, from the recurrence relation (2.4) of the RS polynomial $H_N(x;q)$, we can construct the representation for the Yangian Y(su(n)) invariant motif recursively [8]. The motif was first proposed by Haldane *et al* [3] as eigenstates of the Yangian invariant HS spin chain. The construction of the motifs is as follows; for the N-site spin chain we consider the $N-1$ sequence of '0' and '1'. Here a '0' and a '1' indicates the absence and presence, respectively, of the integer corresponding to the position in the sequence. It is required that n-consecutive 1s do not occur. When the motif is given we can calculate the energies for the HS and the PF chains as

$$E^{\mathrm{HS}} = \sum_j m_j(m_j - N) \tag{2.5}$$

$$E^{\mathrm{PF}} = \sum_j (-m_j) \tag{2.6}$$

where m_j denote a set of positions of 1s. The degeneracy for each motif is given by the 'representation' for the motif. It was pointed out [8] that we can translate the recurrence relation (2.4) for the RS polynomials into one for the motifs. For the su(2) case, we have

$$(\ldots 11) = 0 \tag{2.7a}$$

$$(\ldots 10) = \begin{array}{|c|}\hline \\ \hline\end{array} \otimes (\ldots 1) \tag{2.7b}$$

$$(\ldots 01) = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \otimes (\ldots) \tag{2.7c}$$

$$(\ldots 00) = \begin{array}{|c|}\hline \\ \hline\end{array} \otimes (\ldots 0) - \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \otimes (\ldots). \tag{2.7d}$$

From these identities one can see that the su(2) motif can be decomposed into the elementary motifs (a sequence of '0's) as was pointed out by Haldane *et al* [3]; the su(2) representation is given as a tensor product of each elementary motif, $\underbrace{0\ldots0}_{k} \equiv [k-1]$. On the other hand, for the su(n) case ($n > 2$) the motif cannot be decomposed into the elementary motifs and their recurrence relations for the su(3) case are given as follows

$$(\ldots 111) = 0 \tag{2.8a}$$

$$(\ldots 110) = \begin{array}{|c|}\hline \\ \hline\end{array} \otimes (\ldots 11) \tag{2.8b}$$

$$(\ldots 011) = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} \otimes (\ldots) \tag{2.8c}$$

$$(\ldots 001) = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots 0) - \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots) \tag{2.8d}$$

$$(\ldots 101) = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots 1) \tag{2.8e}$$

$$(\ldots 010) = \begin{array}{|c|}\hline \ \\ \hline \end{array} \otimes (\ldots 01) - \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots 0) \tag{2.8f}$$

$$(\ldots 100) = \begin{array}{|c|}\hline \ \\ \hline \end{array} \otimes (\ldots 10) - \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots 1) \tag{2.8g}$$

$$(\ldots 000) = \begin{array}{|c|}\hline \ \\ \hline \end{array} \otimes (\ldots 00) - \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots 0) + \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \otimes (\ldots). \tag{2.8h}$$

One can see that the ground state of both the HS and the PF su(n) spin chains with $n \times N$ spins is generally given by the motif

$$(\underbrace{11\ldots 1}_{n-1}\,0\,\underbrace{11\ldots 1}_{n-1}\,01\ldots 10\,\underbrace{11\ldots 1}_{n-1}),$$

and that it is non-degenerate.

Recently the relationship between the generalized RS polynomial $H_N(x;q)$ and the Yangian symmetry was re-examined [10, 11]. Another expression of the RS polynomial (2.2) in terms of the 'spinon' basis was given in [10]

$$H_N(x;q) = \sum_{1\leqslant m_1<\cdots<m_s\leqslant N-1} (-)^s \frac{(q;q)_{N-1}}{\prod_{i=1}^{s}(1-q^{-m_i})} s_{[N-m_s]}(x) s_{[m_s-m_{s-1}]}(x)\ldots s_{[m_1]}(x). \tag{2.9}$$

Here m_i denotes a position of a '1' in the su(n) motifs. We shall show that the expression (2.9) satisfies the recurrence relation (2.4) of the RS polynomial. For our purposes, we use the property of the Schur function

$$s_{[1^k]}(x)s_{[m]}(x) = s_{[m+1,1^{k-1}]}(x) + s_{[m,1^k]}(x). \tag{2.10}$$

The proof is as follows. We split the summation in (2.9) into the disjoint union of two cases, the position of $N-1$ is 0 or 1. We thus obtain

$$\begin{aligned} H_N(x;q) = {} & \sum_{1\leqslant m_1<\cdots<m_{s-1}\leqslant N-2} (-)^s \frac{(q;q)_{N-1}}{(1-q^{-(N-1)})\prod_{i=1}^{s-1}(1-q^{-m_i})} \\ & \times s_{[1]}(x)\ldots s_{[N-1-m_{s-1}]}(x)\ldots s_{[m_1]}(x) \\ & + \sum_{1\leqslant m_1<\cdots<m_s\leqslant N-2} (-)^s \frac{(q;q)_{N-1}}{\prod_{i=1}^{s}(1-q^{-m_i})} s_{[N-m_s]}(x)\ldots s_{[m_1]}(x). \end{aligned}$$

By renumbering the position of '1' $\{m_j\}$, we get,

$$\begin{aligned} H_N(x;q) = {} & \sum_{1\leqslant m_1<\cdots<m_s\leqslant N-2} (-)^s \frac{(q;q)_{N-2}}{\prod_{i=1}^{s}(1-q^{-m_i})} \\ & \times (q^{N-1}s_{[1]}(x)\cdot s_{[N-1-m_s]}(x) + (1-q^{N-1})\, s_{[N-m_s]}(x)) \\ & \times s_{[m_s-m_{s-1}]}(x)\ldots s_{[m_1]}(x) \end{aligned}$$

which, due to the property of the Schur function (2.10), reduces to a form

$$H_N(x;q) = s_{[1]}(x)H_{N-1}(x;q) - (1-q^{N-1}) \sum_{1\leqslant m_1<\cdots<m_s\leqslant N-2} (-)^s \frac{(q;q)_{N-2}}{\prod_{i=1}^{s}(1-q^{-m_i})}$$

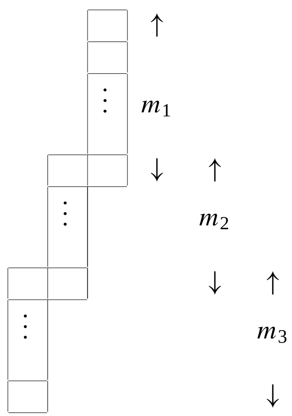
$$\times s_{[N-1-m_s,1]}(x)s_{[m_s-m_{s-1}]}(x)\ldots s_{[m_1]}(x).$$

By applying the property of the Schur function (2.10) recursively, one sees that the expression (2.9) satisfies the same recurrence relation (2.4) with the RS polynomial (2.2). With an initial condition of the 'spinon'-expression (2.9) $H_0(x;q)=1$, we can conclude that the expression (2.9) indeed coincides with the RS polynomial.

The other expression for the RS polynomial (2.2) is proposed in [11] based on the energy function of the path space of the solvable model [14]. The explicit form is given as

$$H_N(x;q)=\sum_{r=1}^{N}\sum_{\substack{m_1+\cdots+m_r=N\\ 1\leqslant m_i\leqslant n}} q^{\frac{1}{2}N(N+1)-\sum_{i=1}^{r}(m_1+\cdots+m_i)}s_{\langle m_1,m_2,\ldots,m_r\rangle}(x). \qquad (2.11)$$

Here $\langle m_1,m_2,\ldots,m_r\rangle$ denotes the border strip of r columns such that the length of the ith column is m_i,



Function $s_{\langle m_1,\ldots,m_r\rangle}(x)$ is the skew Schur function for the border strip $\langle m_1,\ldots,m_r\rangle$, which, due to the Jacobi–Trudi formula, is given as

$$s_{\langle m_1,\ldots,m_r\rangle}(x)=\begin{vmatrix} e_{m_r}(x) & e_{m_r+m_{r-1}}(x) & \ldots & \ldots & \ldots & e_{m_r+\cdots+m_1}(x)\\ 1 & e_{m_{r-1}}(x) & \ldots & \ldots & \ldots & e_{m_{r-1}+\cdots+m_1}(x)\\ & 1 & e_{m_{r-2}}(x) & \ldots & \ldots & e_{m_{r-2}+\cdots+m_1}(x)\\ & & \ddots & \ddots & \vdots & \vdots\\ & & & 1 & e_{m_2}(x) & e_{m_2+m_1}(x)\\ & & & & 1 & e_{m_1}(x)\end{vmatrix}. \qquad (2.12)$$

We see that the skew Schur function $s_{\langle m_1,\ldots,m_r\rangle}(x)$ satisfies the following relation

$$s_{\langle m_1,\ldots,m_r\rangle}(x)=\sum_{k=1}^{r}(-)^{k+1}e_{m_r+\cdots+m_{r-k+1}}(x)s_{\langle m_1,\ldots,m_{r-k}\rangle}(x). \qquad (2.13)$$

Using this identity for the skew Schur function, we can prove that the 'path'-expression (2.11) satisfies the recurrence relation (2.4) of the RS polynomials (see the appendix in [11]). With an initial condition $H_0(x;q)=1$, we can conclude that the expression (2.11) also denotes the RS polynomial (2.2). The 'spinon'-expression (2.9) for the RS polynomial gives us the representation of the motif S as [10]

$$\chi_S(x)=\sum_{S'\subset S}(-)^{s-t}s_{[N-m_{i_t}]}(x)s_{[m_{i_t}-m_{i_{t-1}}]}(x)\ldots s_{[m_{i_1}]}(x) \qquad (2.14)$$

where m_i denote the positions of '1's, $m_1<m_2<\cdots<m_s$ in the motif S. Ordered subsets of S are used, $S'=\{m_{i_1},m_{i_2},\ldots,m_{i_t}\}$. One sees that by definition an expression $\chi_S(x)$ satisfies the recurrence relations, for example, (2.7) and (2.8).

Also the 'path'-expression (2.11) gives the representation for motifs in terms of the skew Schur function. The strategy is as follows; we read motif from the left. We construct the skew Young diagram by adding a box under (respectively left) the box when we encounter '1' (respectively '0') in the motif. We take an example from the $N = 4$ case.

motif S	arrow	skew Young diagram	decomposition
(111)	$\downarrow\downarrow\downarrow$	$\langle 4\rangle$	$[1^4]$
(110)	$\downarrow\downarrow\leftarrow$	$\langle 3, 1\rangle$	$[2, 1^2]$
(101)	$\downarrow\leftarrow\downarrow$	$\langle 2, 2\rangle$	$[2^2] \oplus [2, 1^2]$
(011)	$\leftarrow\downarrow\downarrow$	$\langle 1, 3\rangle$	$[2, 1^2]$
(100)	$\downarrow\leftarrow\leftarrow$	$\langle 2, 1, 1\rangle$	$[3, 1]$
(010)	$\leftarrow\downarrow\leftarrow$	$\langle 1, 2, 1\rangle$	$[3, 1] \oplus [2^2]$
(001)	$\leftarrow\leftarrow\downarrow$	$\langle 1, 1, 2\rangle$	$[3, 1]$
(000)	$\leftarrow\leftarrow\leftarrow$	$\langle 1, 1, 1, 1\rangle$	$[4]$

By definition, the above expression for motifs in terms of the skew Young diagram not only satisfies (2.14) but also the recurrence relation, for example, (2.7) and (2.8).

3. Macdonald polynomial

We have shown that the representation of the Yangian invariant motif can be constructed by use of the recurrence relation (2.4) for the RS polynomials. Mathematically, the RS polynomial is closely related to the Macdonald q-orthogonal polynomial. The Macdonald polynomial $P_\lambda(x; q, t)$ for the Young diagram λ is the eigenfunction of the difference equation [15]

$$\hat{M}_1 P_\lambda(x; q, t) = \left(\sum_{j=1}^{N} t^{N-j} q^{\lambda_j}\right) P_\lambda(x; q, t). \tag{3.1}$$

Here $\hat{M}_1$ is the first-order difference operator defined by

$$\hat{M}_1 = \sum_{j=1}^{n} \left(\prod_{\substack{k=1\\k\neq j}}^{n} \frac{t x_j - x_k}{x_j - x_k}\right) \hat{T}_{q,x_j} \tag{3.2}$$

where $\hat{T}_{q,x_j}$ is the q-difference operator

$$(\hat{T}_{q,x_j} f)(x) = f(x_1, \ldots, q\, x_j, \ldots, x_n)$$

in terms of the affine Hecke algebra, and the generating function was computed [15]. We have generating functions of the Macdonald polynomials $P_\lambda(x; q, t)$. A useful identity among them is for the one-row Macdonald polynomial $P_{[N]}(x; q, t)$

$$\prod_{j=1}^{n} \frac{(t x_j z; q)_\infty}{(x_j z; q)_\infty} = \sum_{N=0}^{\infty} \frac{(t; q)_N}{(q; q)_N} P_{[N]}(x; q, t) z^N. \tag{3.3}$$

Comparing with the generating function for the RS polynomial (2.3), one sees [16] that in the limit $t \to 0$, the Macdonald polynomial reduces to the RS polynomial

$$P_{[N]}(x; q, t = 0) = H_N(x; q). \tag{3.4}$$

See appendix B for the explicit form of the Macdonald polynomials. By applying the q-binomial theorem in the generating function,

$$\sum_{N=0}^{\infty} \frac{(t;q)_N}{(q;q)_N} z^N = \frac{(tz;q)_\infty}{(z;q)_\infty} \tag{3.5}$$

we obtain the explicit form of the Macdonald polynomials $P_{[N]}(x;q,t)$ as

$$P_{[N]}(x;q,t) = \sum_{\substack{k_1+\cdots+k_n=N \\ k_i\geqslant 0}} \frac{(q;q)_N}{(t;q)_N} \frac{(t;q)_{k_1}\ldots(t;q)_{k_n}}{(q;q)_{k_1}\ldots(q;q)_{k_n}} x_1^{k_1}\ldots x_n^{k_n}. \tag{3.6}$$

We find by replacing z with qz in the generating function (3.3) that the one-row Macdonald polynomials $P_{[N]}(x;q,t)$ satisfy the recurrence relation

$$P_{[N]}(x;q,t) = \sum_{k=1}^{n} (-)^{k-1} e_k(x) \frac{(q;q)_{N-1}}{(q;q)_{N-k}} \frac{(t;q)_{N-k}(1-q^{N-k}t^k)}{(t;q)_N} P_{[N-k]}(x;q,t). \tag{3.7}$$

In the following we also use the integral form of the Macdonald polynomial

$$J_{[N]}(x;q,t) = (t;q)_N \cdot P_{[N]}(x;q,t). \tag{3.8}$$

We now consider the one-parameter deformation of the 'spinon'- and 'path'-expressions for the RS polynomials studied in the previous section. First, as a deformation of the 'spinon'-expression (2.9), we propose a form

$$\begin{aligned} P_{[N]}(x;q,t) = &\sum_{1\leqslant m_1<\cdots<m_s\leqslant N-1} (-)^s \frac{(q;q)_{N-1}}{\prod_{i=1}^{s}(1-q^{-m_i})} \frac{(1-t)^{s+1}}{(t;q)_N} \\ &\times P_{[N-m_s]}(x;t) P_{[m_s-m_{s-1}]}(x;t)\ldots P_{[m_1]}(x;t). \end{aligned} \tag{3.9}$$

Here $P_\lambda(x;t)$ is the Hall–Littlewood (HL) polynomial [15] defined by a reduction of the Macdonald polynomials,

$$P_\lambda(x;t) = P_\lambda(x;q=0,t)$$

but, for simple Young diagrams, the HL polynomials are written in terms of the Schur functions $s_\lambda(x)$ as follows:

$$P_{[N]}(x;t) = \sum_{k=0}^{N-1} (-t)^k s_{[N-k,1^k]}(x) \tag{3.10}$$

$$P_{[1^k]}(x;t) = s_{[1^k]}(x) = e_k(x). \tag{3.11}$$

We set for our convenience the integral form of the HL polynomial $P_\lambda(x;t)$ [15] as

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t). \tag{3.12}$$

Here the polynomial $b_\lambda(t)$ is defined by

$$b_\lambda(t) = \prod_{i\geqslant 1} (t;t)_{m_i(\lambda)}$$

where $m_i(\lambda)$ denotes the number of λ_is equal to i. As a one-parameter deformation of an identity (2.10) of the Schur function, we have an identity for the HL polynomials

$$Q_{[r]}(x;t)Q_{[1^s]}(x;t) = (1-t^s)Q_{[r+1,1^{s-1}]}(x;t) + Q_{[r,1^s]}(x;t). \tag{3.13}$$

By use of the above formula, one can see that the 'spinon'-expression (3.9) satisfies the recurrence relation (3.7) for the one-row Macdonald polynomials. The proof is essentially the same as the case of the RS polynomial discussed in the previous section. With an initial condition $P_{[0]}(x;q,t) = 1$, we can conclude that the expression (3.6) coincides exactly with the Macdonald polynomial (3.6), and that it gives the t-deformation of the representation for the Yangian invariant motif (2.9). In conclusion, one sees from (3.9) that, by replacing the Schur functions $s_\lambda(x)$ by the HL polynomials $P_\lambda(x;t)$, the order of $(1-t)$ counts a number of spinons.

In the same way, we shall deform the 'path'-expression (2.11) of the RS polynomial. We propose an expression

$$P_{[N]}(x;q,t) = \sum_{r=1}^{N} (tq^{-1};q^{-1})_{N-r} \frac{(t;q)_r}{(t;q)_N} \sum_{\substack{m_1+\cdots+m_r=N \\ 1\leqslant m_i\leqslant n}} q^{\frac{1}{2}N(N+1)-\sum_{i=1}^{r}(m_1+\cdots+m_i)} s_{\langle m_1,\ldots,m_r\rangle}(x). \tag{3.14}$$

In this case we do not use the skew HL function. By use of an identity of the skew Schur function (2.13), we can show that above expression satisfies the recurrence relation of the Macdonald polynomial (3.7) and that an initial condition $P_{[0]}(x;q,t)=1$ is satisfied. Thus the expression (3.14) also coincides with the definition of the one-row Macdonald polynomial (3.6). We remark that, by comparing the expressions (3.14) and (2.11), the 'weight' $w_r(t,q)$ of each path depends on $r(N-r$ denotes a number of 1s in the motif, i.e. the number of quasi-particles)

$$w_r(t,q) = (tq^{-1};q^{-1})_{N-r}\,(t;q)_r. \tag{3.15}$$

4. Concluding remarks

In this paper we have studied the representation of the Yangian invariant motif. The motif is closely related with the generalized Rogers–Szegö polynomials and we have given three expressions of the RS polynomial,

$$\begin{aligned}
H_N(x;q) &= \sum_{\substack{k_1+\cdots+k_n=N \\ k_i\geqslant 0}} \begin{bmatrix} N \\ k_1,\ldots,k_n \end{bmatrix}_q x_1^{k_1}\ldots x_n^{k_n} \\
&= \sum_{1\leqslant m_1<\cdots<m_s\leqslant N-1} (-)^s \frac{(q;q)_{N-1}}{\prod_{i=1}^{s}(1-q^{-m_i})} s_{[N-m_s]}(x)s_{[m_s-m_{s-1}]}(x)\ldots s_{[m_1]}(x) \\
&= \sum_{r=1}^{N} \sum_{\substack{m_1+\cdots+m_r=N \\ 1\leqslant m_i\leqslant n}} q^{\frac{1}{2}N(N+1)-\sum_{i=1}^{r}(m_1+\cdots+m_r)} s_{\langle m_1,\ldots,m_r\rangle}(x).
\end{aligned}$$

These expressions give us the representation of the Yangian invariant bases 'motif'.

As the Macdonald polynomial for one-row Young diagrams can be regarded as the one-parameter deformation of the RS polynomial $H_N(x;q)$, we have proposed the one-parameter deformation of the previous three expressions. By checking the recurrence relation and the initial condition we have proved that the Macdonald polynomials for the one-row Young diagram can be expressed as follows:

$$\begin{aligned}
J_{[N]}(x;q,t) &= \sum_{\substack{k_1+\cdots+k_n=N \\ k_j\geqslant 0}} (q;q)_N \frac{(t;q)_{k_1}\ldots(t;q)_{k_n}}{(q;q)_{k_1}\ldots(q;q)_{k_n}} x_1^{k_1}\ldots x_n^{k_n} \\
&= \sum_{1\leqslant m_1<\cdots<m_s\leqslant N-1} (-)^s \frac{(q;q)_{N-1}}{\prod_{i=1}^{s}(1-q^{-m_i})} (1-t)^{s+1} P_{[N-m_s]}(x;t)
\end{aligned}$$

$$\times P_{[m_s-m_{s-1}]}(x;t)\dots P_{[m_1]}(x;t)$$
$$= \sum_{r=1}^{N}(tq^{-1};q^{-1})_{N-r}\cdot(t;q)_r \sum_{\substack{m_1+\cdots+m_r=N\\ 1\leqslant m_i\leqslant n}} q^{\frac{1}{2}N(N+1)-\sum_{i=1}^{r}(m_1+\cdots+m_i)} s_{\langle m_1,\dots,m_r\rangle}(x).$$

These expressions show that a deformation parameter t is related to a number of 1s in motifs, i.e. a number of quasi-particles. We do not know whether the Macdonald polynomial for arbitrary Young diagrams may be constructed by these combinatorial methods.

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Appendix A. Rogers–Szegö polynomials

$$\begin{aligned}
H_0(x;q) &= 1\\
H_1(x;q) &= s_{[1]}(x)\\
H_2(x;q) &= s_{[2]}(x) + q s_{[1^2]}(x)\\
H_3(x;q) &= s_{[3]}(x) + q(1+q)s_{[2,1]}(x) + q^3 s_{[1^3]}(x)\\
H_4(x;q) &= s_{[4]}(x) + q(1+q+q^2)s_{[3,1]}(x)q^2(1+q^2)s_{[2^2]}(x)\\
&\quad + q^3(1+q+q^2)s_{[2,1^2]}(x)q^6 s_{[1^4]}(x)\\
H_5(x;q) &= s_{[5]}(x) + q(1+q)(1+q^2)s_{[4,1]}(x)q^2(1+q+q^2+q^3+q^4)s_{[3,2]}(x)\\
&\quad + q^3(1+q^2)(1+q+q^2)s_{[3,1^2]}(x)q^4(1+q+q^2+q^3+q^4)s_{[2^2,1]}(x)\\
&\quad + q^6(1+q)(1+q^2)s_{[2,1^3]}(x)q^{10}s_{[1^5]}(x).
\end{aligned}$$

Appendix B. Macdonald polynomial

$$\begin{aligned}
J_{[0]}(x;q,t) &= 1\\
J_{[1]}(x;q,t) &= (1-t)s_{[1]}(x)\\
J_{[2]}(x;q,t) &= (1-t)(1-tq)s_{[2]}(x) + (1-t)(q-t)s_{[1^2]}(x)\\
J_{[3]}(x;q,t) &= (1-t)(1-tq)(1-tq^2)s_{[3]}(x) + (1+q)(1-t)(1-tq)(q-t)s_{[2,1]}(x)\\
&\quad + (1-t)(t-q)(t-q^2)s_{[1^3]}(x)\\
J_{[4]}(x;q,t) &= (1-t)(1-tq)(1-tq^2)(1-tq^3)s_{[4]}(x)\\
&\quad + (1+q+q^2)(q-t)(1-t)(1-tq)(1-tq^2)s_{[3,1]}(x)\\
&\quad + q(1+q^2)(q-t)(1-t)^2(1-tq)s_{[2^2]}(x)\\
&\quad + (1+q+q^2)(1-t)(1-tq)(q-t)(q^2-t)s_{[2,1^2]}(x)\\
&\quad + (1-t)(q-t)(q^2-t)(q^3-t)s_{[1^4]}(x)\\
J_{[5]}(x;q,t) &= (1-t)(1-tq)(1-tq^2)(1-tq^3)(1-tq^4)s_{[5]}(x)\\
&\quad + (1+q)(1+q^2)(q-t)(1-t)(1-tq)(1-tq^2)(1-tq^3)s_{[4,1]}(x)\\
&\quad + q(1+q+q^2+q^3+q^4)(1-t)^2(q-t)(1-tq)(1-tq^2)s_{[3,2]}(x)\\
&\quad + (1+q^2)(1+q+q^2)(q-t)(q^2-t)(1-t)(1-tq)(1-tq^2)s_{[3,1^2]}(x)
\end{aligned}$$

$$+q(1+q+q^2+q^3+q^4)(1-t)^2(q-t)(q^2-t)(1-tq)s_{[2^2,1]}(x)$$
$$+(1+q)(1+q^2)(q-t)(q^2-t)(q^3-t)(1-t)(1-tq)s_{[2,1^3]}(x)$$
$$+(q-t)(q^2-t)(q^3-t)(q^4-t)(1-t)s_{[1^5]}(x).$$

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